EJAM
www.ejamjournal.com
Electronic Journal of Applied Mathematics
Research article
Mersin University Publishing
DOI: 10.61383/ejam.20231239
2023, Vol. 1, No. 2, 71–86

Optical soliton solutions of the generalized sine-Gordon equation

Yusuf Pandir[1], Humaira Yasmin[2,*]

1. Department of Mathematics, Faculty of Science and Art, Yozgat Bozok University, 66100 Yozgat, Turkey
2. Department of Basic Sciences, Preparatory Year Deanship, King Faisal University, Al-Ahsa 31982 Saudi Arabia

*Corresponding author

Abstract

In this study, the extended trial equation method based on the general form of nonlinear elliptic ordinary differential equation is employed to solve the nonlinear generalized sine-Gordon equations. By the using of this method, we achieve unlike new types of exact wave solutions such as Elliptic-F, Elliptic-E and Elliptic-Π functions that are known as elliptic integrals.

Keywords: trial equation method, elliptic function solution, soliton solution
2020 Mathematics Subject Classification: 35C07, 35C08, 35Q51
Article history: Received 9 Aug 2023; Revised 2 Sep 2023; Accepted 3 Sep 2023; Online 9 Sep 2023

1 Introduction

Recently, the study of nonlinear evolution equations with respect to the mathematical modeling of various physical phenomena has become very important in some physical and engineering applications such as water waves [1], plasma physics [2], nonlinear optics [3], and so on. Various methods [4–16] have been applied to construct the optical soliton solutions to nonlinear differential equations. Examples of some methods that have been used so far are the inverse scattering method, similarity transformation, generalized Jacob elliptic function expansion method, exp-function method, extended F-expansion method, different versions of (G'/G)-expansion method, Kudryashov's method, ansatz method, and the others. Bright optical soliton, dark optical soliton, compactons, singular solitons, doubly-periodic solutions, and other optical solutions have been discovered by use of the above-mentioned methods [4–16]. The optical soliton solutions are very significant and seem in assorted areas of physics, engineering, and applied sciences. Optical solitons are a type of nonlinear wave that provides long-range, high-capacity, and lossless transmission. Therefore, when examined from a physical perspective, they are a special type of soliton, waves that propagate without distortion throughout the

Contact: Yusuf Pandir yusufpandir@gmail.com; Humaira Yasmin hhassain@kfu.edu.sa

propagation dimension. Therefore, it is important to investigate optical soliton solutions of nonlinear evolution equations.

In the recent past, Liu defined the trial equation method in that the elliptic differential equations and the complete discrimination system for polynomials are used [17]. Then, a new version of the trial equation method for the nonlinear problems with rank inhomogeneous is introduced by Liu [18]. A new trial equation method, which is more general than the previous trial equation methods, is proposed in Ref. [19]. Also, Du developed an irrational trial equation method and hence applied this interesting approach to some nonlinear physical problems [20]. A dissimilar trial equation method according to the symmetric features of the differential equation is offered to attain optical soliton solutions to nonlinear differential equations [21]. Apart from these, Pandir et al. constructed the extended trial equation method and tested this powerful method in [22]. The extended trial equation method has many advantages over other trial equation methods. With this method, it is possible to find the rational function solution, the optical singular soliton solution, and the Jacobi elliptic function solutions at the same time. On the other hand, the utility of these methods is illustrated by some applications [23–29].

Our goal in this paper is to investigate the extended trial equation method for the optical soliton solutions to the sine-Gordon equation [30–33]

$$u_{xt} = \sin(nu), \tag{1}$$

where the spatial coordinate x and the temporal coordinate t are the independent variables, u is the dependent variable. Using the transformation

$$u(x,t) = u\left(ax + \frac{t}{a}, ax - \frac{t}{a}\right), \quad a > 0, \tag{2}$$

we can reduce Eq. (1) to the alternative form

$$u_{tt} - u_{xx} + \sin(nu) = 0. \tag{3}$$

The sine-Gordon equation is one of the most important equations in many scientific fields such as the propagation of fluxons in Josephson junctions among two superconductors, the motion of grid pendula attached to a stretched wire, solid state physics, nonlinear optics, and dislocations in metals. In the literature, there have been several species of solutions, including the one-soliton solution, two-soliton solutions, soliton-antisoliton collision, and the breather solution for sine-Gordon equation [33–35]. In this paper, we give the classification of the traveling wave solutions to Eq. (3) for $n = 1$ and $n = 2$, respectively. Thus, we attain some new optical soliton solutions such as singular optical solutions, elliptic integral functions F, E, Π, and Jacobi elliptic function solutions.

2 The extended trial equation method

Step 1. We consider a general form of nonlinear differential equation

$$P(u, u_t, u_x, u_{xx}, \dots) = 0. \tag{4}$$

Under a general form of the wave transformation

$$u(x_1, x_2, \dots, x_N, t) = u(\eta),\ \eta = \lambda\left(\sum_{j=1}^{N} x_j - ct\right)$$

, where $\lambda \neq 0$ and $c \neq 0$, Eq. (4) becomes

$$N(u, u', u'', \dots) = 0. \tag{5}$$

Step 2. From Ref. [6], the general solutions of Eq. (5) are given as

$$u = \sum_{i=0}^{\delta} \tau_i \Theta^i, \tag{6}$$

where

$$(\Theta')^2 = \Lambda(\Theta) = \frac{\Phi(\Theta)}{\Psi(\Theta)} = \frac{\xi_\theta \Theta^\theta + \cdots + \xi_1 \Theta + \xi_0}{\zeta_\epsilon \Theta^\epsilon + \cdots + \zeta_1 \Theta + \zeta_0}. \tag{7}$$

From Eqs. (6) and (7), we find the following relations

$$(u')^2 = \frac{\Phi(\Theta)}{\Psi(\Theta)} \left(\sum_{i=0}^{\delta} i \tau_i \Theta^{i-1} \right)^2, \tag{8}$$

$$u'' = \frac{\Phi'(\Theta)\Psi(\Theta) - \Phi(\Theta)\Psi'(\Theta)}{2\Psi^2(\Theta)} \left(\sum_{i=0}^{\delta} i \tau_i \Theta^{i-1} \right) + \frac{\Phi(\Theta)}{\Psi(\Theta)} \left(\sum_{i=0}^{\delta} i(i-1) \tau_i \Theta^{i-2} \right), \tag{9}$$

where $\Phi(\Theta)$ and $\Psi(\Theta)$ are polynomials. Substituting Eqs. (8) and (9) into Eq. (5), we obtain an algebraic equation of polynomial $\Omega(\Theta)$ of Θ :

$$\Omega(\Theta) = \varrho_s \Theta^s + \cdots + \varrho_1 \Theta + \varrho_0 = 0. \tag{10}$$

A balancing process is applied between the term with the highest order derivative and the term with the highest nonlinearity in Eq. (5). In accordance with the balance principle, we can examine a mathematical relation for θ, ϵ, and δ, and then identify some values of these variables.
Step 3. If the coefficients of $\Omega(\Theta)$ is equal to zero, we build a system of algebraic equations

$$\varrho_i = 0, \qquad i = 0, \ldots, s. \tag{11}$$

The values of $\xi_0, \ldots, \xi_\theta, \zeta_0, \ldots, \zeta_\epsilon$ and $\tau_0, \ldots, \tau_\delta$ can be identified by solving this system algebraically.
Step 4. Eq. (7) can be converted to the following equation by integrating it once:

$$\pm(\eta - \eta_0) = \int \frac{d\Theta}{\sqrt{\Lambda(\Theta)}} = \int \sqrt{\frac{\Psi(\Theta)}{\Phi(\Theta)}} d\Theta. \tag{12}$$

Utilization the roots of $\Phi(\Theta)$, we resolve Eq. (12) by aid of Mathematica software program and categorize the exact approximate solutions to Eq. (4), respectively.

3 Applications to the extended trial equation method

In this section we consider the generalized sine-Gordon equation. In order to apply extended trial equation method, we put forth the transformations

$$v = e^{iu} \tag{13}$$

and therefore

$$\sin u = \frac{v^2 - 1}{2iv}, \quad \cos u = \frac{v^2 + 1}{2v}, \tag{14}$$

and also gives

$$u = \arccos \left[\frac{v^2 + 1}{2v} \right]. \tag{15}$$

Applying this transformation to the generalized sine-Gordon equation, then we have a form

$$2vv_{tt} - 2vv_{xx} - 2v_t^2 + 2v_x^2 + v^{n+2} - v^{-n+2} = 0. \tag{16}$$

In order to demonstrate for traveling wave solutions of Eq. (16), we take the transformation $v(x,t) = V(\eta)$, $\eta = x - ct$, where c as the wave speed. Therefore it can be converted to the ODE

$$2(c^2-1)VV'' - 2(c^2-1)(V')^2 + V^{n+2} - V^{-n+2} = 0, \tag{17}$$

where prime denotes the derivative with respect to η. To identify the parameter n we usually balance the linear terms of the highest order in Eq. (17) with the highest order nonlinear terms.

i) If we take $n = 1$, then Eq. (3) becomes

$$u_{tt} - u_{xx} + \sin u = 0. \tag{18}$$

When we take $n = 1$, then Eq. (17) becomes as follows

$$2(c^2-1)VV'' + 2(1-c^2)(V')^2 + V^3 - V = 0, \tag{19}$$

where prime demonstrates the derivative in accordance with η. Balancing process is applied between the highest order derivative term VV'' and the term with the highest nonlinearity V^3 in Eq. (19). Substituting Eqs. (6) and (7) into Eq. (19) and using balance principle gives

$$\theta = \epsilon + \delta + 2.$$

After this solution method, we acheive the results as follows:

Case 1

If we receive $\epsilon = 0$, $\delta = 1$ and $\theta = 3$, then

$$(v')^2 = \frac{(\tau_1)^2(\xi_3\Theta^3 + \xi_2\Theta^2 + \xi_1\Theta + \xi_0)}{\zeta_0}, \tag{20}$$

$$v'' = \frac{\tau_1(3\xi_3\Theta^2 + 2\xi_2\Theta + \xi_1)}{2\zeta_0}, \tag{21}$$

where $\xi_3 \neq 0$, $\zeta_0 \neq 0$. Serially, solving the algebraic equation system (11) gives

$$\xi_0 = \frac{\tau_0\left(\xi_1\tau_1^2 - \xi_3(\tau_0^2-1)\right)}{2\tau_1^3}, \xi_1 = \xi_1, \xi_2 = \frac{\xi_3\left(3\tau_0^2-1\right) + \xi_1\tau_1^2}{2\tau_0\tau_1},$$

$$\xi_3 = \xi_3, \quad \tau_0 = \tau_0, \quad \tau_1 = \tau_1, \quad \zeta_0 = \zeta_0, c = \pm\sqrt{\frac{\xi_3 - \zeta_0\tau_1}{\xi_3}}. \tag{22}$$

Substituting these consequences into Eqs. (6) and (12), we get

$$\pm(\eta - \eta_0) = \sqrt{\frac{\zeta_0}{\xi_3}} \int \frac{d\Theta}{\sqrt{\Theta^3 + \frac{(3\tau_0^2-1)+\xi_1\tau_1^2}{2\tau_0\tau_1}\Theta^2 + \frac{\xi_1}{\xi_3}\Theta + \frac{\tau_0\left(\xi_1\tau_1^2 - \xi_3(\tau_0^2-1)\right)}{2\xi_3\tau_1^3}}}. \tag{23}$$

Integrating Eq. (23), we acquire the solutions to the Eq. (18) as follows:

$$\pm(\eta - \eta_0) = -\frac{2A}{\sqrt{\Theta - \alpha_1}}, \tag{24}$$

$$\pm(\eta-\eta_0)=\frac{2A}{\sqrt{\alpha_2-\alpha_1}}\arctan\sqrt{\frac{\Theta-\alpha_2}{\alpha_2-\alpha_1}},\quad \alpha_2>\alpha_1, \tag{25}$$

$$\pm(\eta-\eta_0)=\frac{A}{\sqrt{\alpha_1-\alpha_2}}\ln\left|\frac{\sqrt{\Theta-\alpha_2}-\sqrt{\alpha_1-\alpha_2}}{\sqrt{\Theta-\alpha_2}+\sqrt{\alpha_1-\alpha_2}}\right|,\quad \alpha_1>\alpha_2, \tag{26}$$

$$\pm(\eta-\eta_0)=\frac{2A}{\sqrt{\alpha_1-\alpha_3}}F(\varphi,l),\quad \alpha_1>\alpha_2>\alpha_3, \tag{27}$$

where

$$A=\sqrt{\frac{\zeta_0}{\xi_3}},\quad F(\varphi,l)=\int_0^{\varphi}\frac{d\psi}{\sqrt{1-l^2\sin^2\psi}}, \tag{28}$$

and

$$\varphi=\arcsin\sqrt{\frac{\Theta-\alpha_3}{\alpha_2-\alpha_3}},\quad l^2=\frac{\alpha_2-\alpha_3}{\alpha_1-\alpha_3}. \tag{29}$$

Also α_1, α_2, and α_3 are the roots of the polynomial equation

$$\Theta^3+\frac{\xi_2}{\xi_3}\Theta^2+\frac{\xi_1}{\xi_3}\Theta+\frac{\xi_0}{\xi_3}=0. \tag{30}$$

Substituting the solutions (24)-(27) into (6) and (15), we obtain

$$u(x,t)=\arccos\left[\frac{1}{2}\frac{\left(\tau_0+\tau_1\alpha_1+\tau_1\frac{4A^2}{\left(x\pm\sqrt{\frac{\xi_3-\zeta_0\tau_1}{\xi_3}}t-\eta_0\right)}\right)^2+1}{\tau_0+\tau_1\alpha_1+\tau_1\frac{4A^2}{\left(x\pm\sqrt{\frac{\xi_3-\zeta_0\tau_1}{\xi_3}}t-\eta_0\right)}}\right], \tag{31}$$

$$u(x,t)=\arccos\left[\frac{\left(\varsigma+\tau_1(\alpha_2-\alpha_1)\tanh^2\left(\frac{\sqrt{\alpha_1-\alpha_2}}{A}\eta\right)\right)^2+1}{2\left(\varsigma+\tau_1(\alpha_2-\alpha_1)\tanh^2\left(\frac{\sqrt{\alpha_1-\alpha_2}}{A}\eta\right)\right)}\right], \tag{32}$$

$$u(x,t)=\arccos\left[\frac{\left(\varsigma+\tau_1(\alpha_1-\alpha_2)\operatorname{cosech}^2\left(\frac{\sqrt{\alpha_1-\alpha_2}}{A}\eta\right)\right)^2+1}{2\left(\varsigma+\tau_1(\alpha_1-\alpha_2)\operatorname{cosech}^2\left(\frac{\sqrt{\alpha_1-\alpha_2}}{A}\eta\right)\right)}\right], \tag{33}$$

$$u(x,t)=\arccos\left[\frac{1}{2}\frac{\left(\varrho+\tau_1(\alpha_2-\alpha_3)\mathbf{sn}^2\left(\pm\frac{\sqrt{\alpha_1-\alpha_3}}{A}\eta,l^2\right)\right)^2+1}{\varrho+\tau_1(\alpha_2-\alpha_3)\mathbf{sn}^2\left(\pm\frac{\sqrt{\alpha_1-\alpha_3}}{A}\eta,l^2\right)}\right], \tag{34}$$

where $\eta=\left(x\pm\sqrt{\frac{\xi_3-\zeta_0\tau_1}{\xi_3}}t-\eta_0\right)$, $\varsigma=\tau_0+\tau_1\alpha_1$, $\varrho=\tau_0+\tau_1\alpha_3$ and $l^2=\frac{\alpha_2-\alpha_3}{\alpha_1-\alpha_3}$.

Remark 1. The optical soliton solutions (31)-(34) acquired by use of the extended trial equation method for Eq. (18) have been checked up on Mathematica software program. According to our determination, the rational function solution, the optical singular soliton solution and the Jacobi elliptic function solutions, that we obtain in this paper, are not demonstrated in the previous literature. These results are new optical solutions of Eq. (18).

Case 2

Figure 1: Optical solution of Eq.(3.19) is demonstrated at $\tau_0 = \frac{1}{2}, \quad \tau_1 = 1, \quad \alpha_1 = \frac{1}{2}, \quad \zeta_0 = 1, \quad \eta_0 = 0,$ $\xi_3 = 2$ with imajinary and real parts respectively.

Figure 2: The graphs indicate the approximate solution of Eq. (3.19) for $t = 1$.

Figure 3: Optical solution of Eq.(3.20) is demonstrated at $\tau_0 = \frac{1}{2}, \quad \tau_1 = 1, \quad \alpha_1 = \frac{1}{2}, \quad \alpha_2 = 1, \quad \zeta_0 = 1,$ $\eta_0 = 0, \quad \xi_3 = 2$ and second graph indicates the approximate solution for $t = 1$.

Figure 4: Optical solution of Eq.(3.21) is demonstrated at $\tau_0 = \frac{1}{2}, \quad \tau_1 = \frac{1}{2}, \quad \alpha_1 = 1, \quad \alpha_2 = \frac{1}{2}, \quad \zeta_0 = 1,$ $\eta_0 = 0, \quad \xi_3 = 2$ and second graph indicates the approximate solution for $t = 1$.

Figure 5: Optical solution of Eq.(3.22) is demonstrated at $\tau_0 = \frac{1}{2}, \quad \tau_1 = 1, \quad \alpha_1 = \frac{1}{2}, \quad \alpha_2 = 1, \quad \zeta_0 = 1,$ $\eta_0 = 0, \quad \xi_3 = 2$ and second graph indicates the approximate solution for $t = 1$.

If we get $\epsilon = 0$, $\delta = 2$ and $\theta = 4$, then

$$(v')^2 = \frac{(\tau_1 + 2\tau_2\Theta)^2(\xi_4\Theta^4 + \xi_3\Theta^3 + \xi_2\Theta^2 + \xi_1\Theta + \xi_0)}{\zeta_0}, \tag{35}$$

$$v'' = \frac{(\tau_1+2\tau_1\Theta)(4\xi_4\Theta^3+3\xi_3\Theta^2+2\xi_2\Theta+\xi_1)}{2\zeta_0} + \frac{2\tau_2(\xi_4\Theta^4+\xi_3\Theta^3+\xi_2\Theta^2+\xi_1\Theta+\xi_0)}{\zeta_0}, \tag{36}$$

where $\xi_4 \neq 0$, $\zeta_0 \neq 0$. Respectively, solving the algebraic equation system (11) yields as follows:

$$\xi_0 = \frac{8\xi_2\tau_1^3\tau_2 + \xi_3\left(16\tau_2^2 - 5\tau_1^4\right)}{32\tau_1\tau_2^3},\ \xi_1 = -\frac{\tau_1\left(\xi_3\tau_1 - 2\xi_2\tau_2\right)}{2\tau_2^2},\ \xi_2 = \xi_2,\ \xi_3 = \xi_3,$$

$$\xi_4 = \frac{\xi_3\tau_2}{2\tau_1},\ \zeta_0 = \zeta_0,\ \tau_0 = \frac{\tau_1^2}{4\tau_2},\ \tau_1 = \tau_1,\ \tau_2 = \tau_2,\ c = \pm\sqrt{\frac{2\xi_3 - \zeta_0\tau_1}{2\xi_3}}. \tag{37}$$

Substituting these consequences into Eqs. (6) and (12), we acquire

$$\pm(\eta - \eta_0) = \sqrt{\frac{2\zeta_0\tau_1}{\xi_3\tau_2}} \int \frac{d\Theta}{\sqrt{\Theta^4 + \frac{2\tau_1}{\tau_2}\Theta^3 + \frac{2\xi_2\tau_1}{\xi_3\tau_2}\Theta^2 - \frac{\tau_1^2\left(\xi_3\tau_1 - 2\xi_2\tau_2\right)}{\xi_3\tau_2^3}\Theta + \frac{8\xi_2\tau_1^3\tau_2 + \xi_3\left(16\tau_2^2 - 5\tau_1^4\right)}{16\xi_3\tau_2^4}}}. \tag{38}$$

Integrating Eq. (38), we get the solutions to the Eq. (18) as follows:

$$\pm(\eta - \eta_0) = -\frac{A}{\Theta - \alpha_1}, \tag{39}$$

$$\pm(\eta - \eta_0) = \frac{2A}{\alpha_1 - \alpha_2}\sqrt{\frac{\Theta - \alpha_2}{\Theta - \alpha_1}}, \quad \alpha_1 > \alpha_2, \tag{40}$$

$$\pm(\eta - \eta_0) = \frac{A}{\alpha_1 - \alpha_2}\ln\left|\frac{\Theta - \alpha_1}{\Theta - \alpha_2}\right|, \tag{41}$$

$$\pm(\eta - \eta_0) = \frac{2A}{\sqrt{(\alpha_1-\alpha_2)(\alpha_1-\alpha_3)}}\ln\left|\frac{\sqrt{(\Theta-\alpha_2)(\alpha_1-\alpha_3)} - \sqrt{(\Theta-\alpha_3)(\alpha_1-\alpha_2)}}{\sqrt{(\Theta-\alpha_2)(\alpha_1-\alpha_3)} + \sqrt{(\Theta-\alpha_3)(\alpha_1-\alpha_2)}}\right|, \quad \alpha_1 > \alpha_2 > \alpha_3, \tag{42}$$

$$\pm(\eta - \eta_0) = \frac{2A}{\sqrt{(\alpha_1 - \alpha_3)(\alpha_2 - \alpha_4)}}F(\varphi, l), \quad \alpha_1 > \alpha_2 > \alpha_3 > \alpha_4, \tag{43}$$

where

$$A = \sqrt{\frac{2\zeta_0\tau_1}{\xi_3\tau_2}}, \quad \varphi = \arcsin\sqrt{\frac{(\Theta-\alpha_1)(\alpha_2-\alpha_4)}{(\Theta-\alpha_2)(\alpha_1-\alpha_4)}}, \quad l^2 = \frac{(\alpha_2-\alpha_3)(\alpha_1-\alpha_4)}{(\alpha_1-\alpha_3)(\alpha_2-\alpha_4)}. \tag{44}$$

Also α_1, α_2, α_3 and α_4 are the roots of the polynomial equation

$$\Theta^4 + \frac{\xi_3}{\xi_4}\Theta^3 + \frac{\xi_2}{\xi_4}\Theta^2 + \frac{\xi_1}{\xi_4}\Theta + \frac{\xi_0}{\xi_4} = 0. \tag{45}$$

Replacing the optical solutions (39)-(43) into (6) and (15), we get

$$u(x,t) = \arccos\left[\frac{1}{2}\frac{\left(\tau_0+\tau_1\alpha_1 \pm \frac{\tau_1 A}{x \pm \sqrt{\frac{2\xi_3-\zeta_0\tau_1}{2\xi_3}}t-\eta_0} + \tau_2\left(\alpha_1 \pm \frac{A}{x \pm \sqrt{\frac{2\xi_3-\zeta_0\tau_1}{2\xi_3}}t-\eta_0}\right)^2\right)^2 + 1}{\tau_0+\tau_1\alpha_1 \pm \frac{\tau_1 A}{x \pm \sqrt{\frac{2\xi_3-\zeta_0\tau_1}{2\xi_3}}t-\eta_0} + \tau_2\left(\alpha_1 \pm \frac{A}{x \pm \sqrt{\frac{2\xi_3-\zeta_0\tau_1}{2\xi_3}}t-\eta_0}\right)^2}\right], \tag{46}$$

Figure 6: Optical solution of Eq.(3.44) is demonstrated at $\tau_0=\frac{1}{2}$, $\tau_1=1$, $\tau_2=1$, $\alpha_1=\frac{1}{2}$, $\zeta_0=1$, $\eta_0=0$, $\xi_3=2$ and second graph indicates the approximate solution for $t=1$.

$$u(x,t)=\arccos\left[\frac{1}{2}\frac{\left(\tau_0+\tau_1\alpha_1+\frac{4A^2(\alpha_2-\alpha_1)\tau_1}{4A^2-((\alpha_1-\alpha_2)\eta)^2}+\tau_2\left(\alpha_1+\frac{4A^2(\alpha_2-\alpha_1)}{4A^2-((\alpha_1-\alpha_2)\eta)^2}\right)^2\right)^2+1}{\tau_0+\tau_1\alpha_1+\frac{4A^2(\alpha_2-\alpha_1)\tau_1}{4A^2-((\alpha_1-\alpha_2)\eta)^2}+\tau_2\left(\alpha_1+\frac{4A^2(\alpha_2-\alpha_1)}{4A^2-((\alpha_1-\alpha_2)\eta)^2}\right)^2}\right], \quad (47)$$

$$u(x,t)=\arccos\left[\frac{1}{2}\frac{\left(\tau_0+\tau_1\alpha_2+\frac{(\alpha_2-\alpha_1)\tau_1}{\exp\left(\frac{\alpha_1-\alpha_2}{A}\eta\right)-1}+\tau_2\left(\alpha_2+\frac{(\alpha_2-\alpha_1)}{\exp\left(\frac{\alpha_1-\alpha_2}{A}\eta\right)-1}\right)^2\right)^2+1}{\tau_0+\tau_1\alpha_2+\frac{(\alpha_2-\alpha_1)\tau_1}{\exp\left(\frac{\alpha_1-\alpha_2}{A}\eta\right)-1}+\tau_2\left(\alpha_2+\frac{(\alpha_2-\alpha_1)}{\exp\left(\frac{\alpha_1-\alpha_2}{A}\eta\right)-1}\right)^2}\right], \quad (48)$$

$$u(x,t)=\arccos\left[\frac{1}{2}\frac{\left(\tau_0+\tau_1\alpha_1+\frac{(\alpha_1-\alpha_2)\tau_1}{\exp\left(\frac{\alpha_1-\alpha_2}{A}\eta\right)-1}+\tau_2\left(\alpha_1+\frac{(\alpha_1-\alpha_2)}{\exp\left(\frac{\alpha_1-\alpha_2}{A}\eta\right)-1}\right)^2\right)^2+1}{\tau_0+\tau_1\alpha_1+\frac{(\alpha_1-\alpha_2)\tau_1}{\exp\left(\frac{\alpha_1-\alpha_2}{A}\eta\right)-1}+\tau_2\left(\alpha_1+\frac{(\alpha_1-\alpha_2)}{\exp\left(\frac{\alpha_1-\alpha_2}{A}\eta\right)-1}\right)^2}\right], \quad (49)$$

$$u(x,t)=\arccos\left[\frac{1}{2}\frac{\left(\tau_0+\tau_1\alpha_1-\frac{\widetilde{A}\tau_1}{B+(\alpha_3-\alpha_2)\cosh(C\eta)}+\tau_2\left(\alpha_1-\frac{\widetilde{A}}{B+(\alpha_3-\alpha_2)\cosh(C\eta)}\right)^2\right)^2+1}{\tau_0+\tau_1\alpha_1-\frac{\widetilde{A}\tau_1}{B+(\alpha_3-\alpha_2)\cosh(C\eta)}+\tau_2\left(\alpha_1-\frac{\widetilde{A}}{B+(\alpha_3-\alpha_2)\cosh(C\eta)}\right)^2}\right], \quad (50)$$

where $\widetilde{A}=2(\alpha_1-\alpha_2)(\alpha_1-\alpha_3)$, $B=2\alpha_1-\alpha_2-\alpha_3$, $C=\frac{\sqrt{(\alpha_1-\alpha_2)(\alpha_1-\alpha_3)}}{2A}$, $\eta=x\pm\sqrt{\frac{2\xi_3-\zeta_0\tau_1}{2\xi_3}}t-\eta_0$.

$$u(x,t)=\arccos\left[\frac{1}{2}\frac{\left(\tau_0+\tau_1\alpha_2+\frac{D\tau_1}{E\mathrm{sn}^2(\varphi,l^2)+\alpha_4-\alpha_2}+\tau_2\left(\alpha_2+\frac{D\tau_1}{E\mathrm{sn}^2(\varphi,l^2)+\alpha_4-\alpha_2}\right)^2\right)^2+1}{\tau_0+\tau_1\alpha_2+\frac{D\tau_1}{E\mathrm{sn}^2(\varphi,l^2)+\alpha_4-\alpha_2}+\tau_2\left(\alpha_2+\frac{D\tau_1}{E\mathrm{sn}^2(\varphi,l^2)+\alpha_4-\alpha_2}\right)^2}\right], \quad (51)$$

where

$$D=(\alpha_1-\alpha_2)(\alpha_4-\alpha_2),\ E=\alpha_1-\alpha_4,$$

and

$$\varphi=\frac{\sqrt{(\alpha_1-\alpha_3)(\alpha_2-\alpha_4)}}{2A}\left(x\pm\sqrt{\frac{2\xi_3-\zeta_0\tau_1}{2\xi_3}}t-\eta_0\right), l^2=\frac{(\alpha_2-\alpha_3)(\alpha_1-\alpha_4)}{(\alpha_1-\alpha_3)(\alpha_2-\alpha_4)}$$

For plain display, if we get $\eta_0=0$, then we can rewrite the optical soliton solutions (46)-(51) as follows:

$$u(x,t)=\arccos\left(\frac{1}{2}\sum_{i=0}^{2}\frac{\left(\tau_i\left(\alpha_1\pm\frac{A}{x-vt}\right)^i\right)^2+1}{\tau_i\left(\alpha_1\pm\frac{A}{x-vt}\right)^i}\right), \tag{52}$$

$$u(x,t)=\arccos\left(\frac{1}{2}\sum_{i=0}^{2}\frac{\left(\tau_i\left(\alpha_1+\frac{4A^2(\alpha_1-\alpha_2)}{4A^2-[(\alpha_1-\alpha_2)(x-vt)]^2}\right)^i\right)^2+1}{\tau_i\left(\alpha_1+\frac{4A^2(\alpha_1-\alpha_2)}{4A^2-[(\alpha_1-\alpha_2)(x-vt)]^2}\right)^i}\right), \tag{53}$$

$$u(x,t)=\arccos\left(\frac{1}{2}\sum_{i=0}^{2}\frac{\left(\tau_i\left(\alpha_2+\frac{\alpha_2-\alpha_1}{\exp[B_3(x-vt)]-1}\right)^i\right)^2+1}{\tau_i\left(\alpha_2+\frac{\alpha_2-\alpha_1}{\exp[B_3(x-vt)]-1}\right)^i}\right), \tag{54}$$

$$u(x,t)=\arccos\left(\frac{1}{2}\sum_{i=0}^{2}\frac{\left(\tau_i\left(\alpha_1+\frac{\alpha_1-\alpha_2}{\exp[B_3(x-vt)]-1}\right)^i\right)^2+1}{\tau_i\left(\alpha_1+\frac{\alpha_1-\alpha_2}{\exp[B_3(x-vt)]-1}\right)^i}\right), \tag{55}$$

$$u(x,t)=\arccos\left(\frac{1}{2}\sum_{i=0}^{2}\frac{\left(\tau_i\left(\alpha_1-\frac{\widetilde{A}}{B+(\alpha_3-\alpha_2)\cosh[C(x-vt)]}\right)^i\right)^2+1}{\tau_i\left(\alpha_1-\frac{\widetilde{A}}{B+(\alpha_3-\alpha_2)\cosh[C(x-vt)]}\right)^i}\right), \tag{56}$$

$$u(x,t)=\arccos\left(\frac{1}{2}\sum_{i=0}^{2}\frac{\left(\tau_i\left(\alpha_2+\frac{D}{\alpha_4-\alpha_2+F\mathbf{sn}^2(\varphi,l)}\right)^i\right)^2+1}{\tau_i\left(\alpha_2+\frac{D}{\alpha_4-\alpha_2+F\mathbf{sn}^2(\varphi,l)}\right)^i}\right), \tag{57}$$

where

$$B_3=\frac{(\alpha_1-\alpha_2)}{A},\ \varphi=\frac{\sqrt{(\alpha_1-\alpha_3)(\alpha_2-\alpha_4)}}{2A}(x-vt),\ l^2=\frac{(\alpha_2-\alpha_3)(\alpha_1-\alpha_4)}{(\alpha_1-\alpha_3)(\alpha_2-\alpha_4)},\ v=\pm\sqrt{\frac{2\xi_3-\zeta_0\tau_1}{2\xi_3}}.$$

Expressed here, A is the amplitude of the soliton, while v is the velocity and B and C are the inverse width of the solitons.

Remark 2. The optical soliton solutions (52)-(57) obtained by use of the suggested method for Eq. (18) have been checked by Mathematica package program. According to our findings, the rational function solution, the optical singular soliton, bright optical soliton, dark optical soliton, and the Jacobi elliptic function solutions, that we find in this article, are not indicated in the previous literature. These obtained results are new optical soliton solutions of Eq. (18).

Case 3: If we take $\epsilon=0$, $\delta=3$ and $\theta=5$, then

$$(v')^2=\frac{(\tau_1+2\tau_2\Theta+3\tau_3\Theta^2)^2(\xi_5\Theta^5+\xi_4\Theta^4+\xi_3\Theta^3+\xi_2\Theta^2+\xi_1\Theta+\xi_0)}{\zeta_0}, \tag{58}$$

$$v''=\frac{2(2\tau_2+6\tau_3\Theta)(\xi_5\Theta^5+\xi_4\Theta^4+\xi_3\Theta^3+\xi_2\Theta^2+\xi_1\Theta+\xi_0)+(\tau_1+2\tau_2\Theta+3\tau_3\Theta^2)(5\xi_5\Theta^4+4\xi_4\Theta^3+3\xi_3\Theta^2+2\xi_2\Theta+\xi_1)}{2\zeta_0}, \tag{59}$$

where $\xi_5 \neq 0$, $\zeta_0 \neq 0$. Respectively, solving the algebraic equation system (11) yields as follows:

$$\xi_0 = \frac{\xi_5\tau_2^2\left(\tau_2^2 - 27\tau_3^2\right)}{243\tau_3^5},\ \xi_1 = \frac{\xi_5\tau_2\left(5\tau_2^2 - 54\tau_3^2\right)}{81\tau_3^4},\ \xi_2 = \frac{\xi_5\left(10\tau_2^2 - 27\tau_3^2\right)}{27\tau_3^3},$$
$$\xi_3 = \frac{10\xi_5\tau_2^2}{9\tau_3^2},\ \xi_4 = \frac{5\xi_5\tau_2}{3\tau_3},\ \xi_5 = \xi_5,\ \zeta_0 = \zeta_0,\ \tau_0 = \frac{-27\tau_3^2 + \tau_2^3}{27\tau_3^2},$$
$$\tau_1 = \frac{\tau_2^2}{3\tau_3},\quad \tau_2 = \tau_2,\quad \tau_3 = \tau_3,\ c = \pm\frac{1}{3}\sqrt{\frac{9\xi_5 - \zeta_0\tau_3}{\xi_5}}. \tag{60}$$

Substituting these consequences into Eqs. (6) and (12), we obtain

$$\pm(\eta - \eta_0) = \sqrt{\frac{\zeta_0}{\xi_5}}\int \frac{d\Theta}{\sqrt{\Theta^5 + \frac{\xi_4}{\xi_5}\Theta^4 + \frac{\xi_3}{\xi_5}\Theta^3 + \frac{\xi_2}{\xi_5}\Theta^2 + \frac{\xi_1}{\xi_5}\Theta + \frac{\xi_0}{\xi_5}}}. \tag{61}$$

Integrating Eq. (61), we get the solutions to the Eq. (18) as follows:

$$\pm(\eta - \eta_0) = -\frac{2A}{3\sqrt{(\Theta - \alpha_1)^3}}, \tag{62}$$

$$\pm(\eta - \eta_0) = \frac{A}{(\alpha_1 - \alpha_2)^{\frac{3}{2}}}\operatorname{arctanh}\left[\sqrt{\frac{\Theta - \alpha_2}{\alpha_1 - \alpha_2}}\right] - \frac{3A\sqrt{\Theta - \alpha_2}}{(\alpha_1 - \alpha_2)(\Theta - \alpha_1)},\ \alpha_1 > \alpha_2, \tag{63}$$

$$\pm(\eta - \eta_0) = -\frac{2A}{(\alpha_1 - \alpha_2)^{\frac{3}{2}}}\arctan\left[\sqrt{\frac{\Theta - \alpha_1}{\alpha_1 - \alpha_2}}\right] - \frac{6A}{\sqrt{\Theta - \alpha_1}(\alpha_1 - \alpha_2)}, \tag{64}$$

$$\pm(\eta - \eta_0) = \frac{2A}{\alpha_1 - \alpha_2}\operatorname{arctanh}\left[\sqrt{\frac{\Theta - \alpha_3}{\alpha_2 - \alpha_3}}\right]\left(\frac{1}{\sqrt{\alpha_2 - \alpha_3}} - \frac{1}{\sqrt{\alpha_1 - \alpha_3}}\right),\ \alpha_1 > \alpha_2 > \alpha_3, \tag{65}$$

$$\pm(\eta - \eta_0) = \frac{-2A}{\sqrt{\Theta - \alpha_1}(\alpha_1 - \alpha_2)(\alpha_1 - \alpha_3)}\left[\sqrt{(\Theta - \alpha_2)(\Theta - \alpha_3)} + i\left(E(\varphi, l) - F(\varphi, l)\right)\right], \tag{66}$$

where

$$A = \sqrt{\frac{\zeta_0}{\xi_5}},\ E(\varphi, l) = \int_0^{\varphi}\sqrt{1 - l^2\sin^2\psi}\,d\psi,\ \varphi = -\arcsin\sqrt{\frac{\Theta - \alpha_1}{\alpha_2 - \alpha_1}},\ l^2 = \frac{\alpha_1 - \alpha_2}{\alpha_1 - \alpha_3},$$
$$\pm(\eta - \eta_0) = \frac{-2iA}{\sqrt{\alpha_2 - \alpha_3}(\alpha_1 - \alpha_2)}\left(F(\varphi, l) - \pi(\varphi, n, l)\right),\ \alpha_1 > \alpha_2 > \alpha_3 > \alpha_4,$$
$$\pi(\varphi, n, l) = \int_0^{\varphi}\frac{d\psi}{(1 + n\sin^2\psi)\sqrt{1 - l^2\sin^2\psi}},\ \varphi = -\arcsin\sqrt{\frac{\alpha_3 - \alpha_2}{\Theta - \alpha_2}},\ l^2 = \frac{\alpha_2 - \alpha_4}{\alpha_2 - \alpha_3},\ n = \frac{\alpha_2 - \alpha_1}{\alpha_2 - \alpha_3}.$$

Also α_1, α_2, α_3, α_4 and α_5 are the roots of the polynomial equation

$$\Theta^5 + \frac{\xi_4}{\xi_5}\Theta^4 + \frac{\xi_3}{\xi_5}\Theta^3 + \frac{\xi_2}{\xi_5}\Theta^2 + \frac{\xi_1}{\xi_5}\Theta + \frac{\xi_0}{\xi_5} = 0. \tag{67}$$

ii) If we get $n = 2$, then Eq. (1) becomes

$$u_{tt} - u_{xx} + \sin(2u) = 0. \tag{68}$$

In this section, we take into account the generalized sine-Gordon equation. Using the transformation (15) and therefore Eq. (4) carry into a ODE form

$$2(c^2 - 1)VV'' + 2(1 - c^2)(V')^2 + V^4 - 1 = 0. \tag{69}$$

Substituting Eqs. (8) and (9) into Eq. (69) and using balance principle gives

$$\theta = \epsilon + 2\delta + 2.$$

According to this solution method, we attain the optical soliton solutions as follows:

Case 1

If we take $\epsilon = 0$, $\delta = 1$ and $\theta = 4$, then

$$(v')^2 = \frac{(\tau_1)^2(\xi_4\Theta^4 + \xi_3\Theta^3 + \xi_2\Theta^2 + \xi_1\Theta + \xi_0)}{\zeta_0}, \tag{70}$$

$$v'' = \frac{\tau_1(4\xi_4\Theta^3 + 3\xi_3\Theta^2 + 2\xi_2\Theta + \xi_1)}{2\zeta_0}, \tag{71}$$

where $\xi_4 \neq 0$, $\zeta_0 \neq 0$. Respectively, solving the equation system (13) gives

$$\xi_0 = \frac{\xi_4 - 5\xi_4\tau_0^4 + \xi_2\tau_0^2\tau_1^2}{\tau_1^4},\ \xi_1 = \frac{2\left(\xi_2\tau_0\tau_1^2 - 4\xi_4\tau_0^3\right)}{\tau_1^3},\ \xi_2 = \xi_2,\ \xi_3 = \frac{4\xi_4\tau_0}{\tau_1},$$

$$\xi_4 = \xi_4,\ \tau_0 = \tau_0,\ \tau_1 = \tau_1,\ \zeta_0 = \zeta_0,\ c = \pm\sqrt{\frac{2\xi_4 - \zeta_0\tau_1^2}{2\xi_4}}. \tag{72}$$

Substituting these consequences into Eqs. (9) and (14), we attain

$$\pm(\eta - \eta_0) = \sqrt{\frac{\zeta_0}{\xi_4}}\int \frac{d\Theta}{\sqrt{\Theta^4 + \frac{4\tau_0}{\xi_3\tau_1}\Theta^3 + \frac{4\tau_0}{\xi_3\tau_1}\Theta^2 + \frac{8\tau_0^2(-\xi_3\tau_0+\xi_2\tau_1)}{\xi_3\tau_1^3}\Theta + \frac{4\tau_0^3(-5\xi_3\tau_0+4\xi_2\tau_1)}{4\xi_3\tau_1^4}}}. \tag{73}$$

Integrating Eq. (73), we get the solutions to the Eq. (68) as follows:

$$\pm(\eta - \eta_0) = \frac{-A}{\Theta - \alpha_1}, \tag{74}$$

$$\pm(\eta - \eta_0) = \frac{2A}{\alpha_1 - \alpha_2}\sqrt{\frac{\Theta - \alpha_2}{\Theta - \alpha_1}},\quad \alpha_2 > \alpha_1, \tag{75}$$

$$\pm(\eta - \eta_0) = \frac{A}{\alpha_1 - \alpha_2}\ln\left|\frac{\Theta - \alpha_1}{\Theta - \alpha_2}\right|,\quad \alpha_1 > \alpha_2, \tag{76}$$

$$\pm(\eta - \eta_0) = \frac{A}{\sqrt{(\alpha_1-\alpha_2)(\alpha_1-\alpha_3)}}\ln\left|\frac{\sqrt{(\Theta-\alpha_2)(\alpha_1-\alpha_3)} - \sqrt{(\Theta-\alpha_3)(\alpha_1-\alpha_2)}}{\sqrt{(\Theta-\alpha_2)(\alpha_1-\alpha_3)} + \sqrt{(\Theta-\alpha_3)(\alpha_1-\alpha_2)}}\right|,\ \alpha_1 > \alpha_2 > \alpha_3, \tag{77}$$

$$\pm(\eta - \eta_0) = \frac{2A}{\sqrt{(\alpha_1 - \alpha_3)(\alpha_2 - \alpha_4)}}F(\varphi, l),\quad \alpha_1 > \alpha_2 > \alpha_3 > \alpha_4, \tag{78}$$

where

$$A = \sqrt{\frac{\zeta_0}{\xi_4}},\quad \varphi = \arcsin\sqrt{\frac{(\Theta - \alpha_1)(\alpha_2 - \alpha_4)}{(\Theta - \alpha_2)(\alpha_1 - \alpha_4)}},\quad l^2 = \frac{(\alpha_2 - \alpha_3)(\alpha_1 - \alpha_4)}{(\alpha_1 - \alpha_3)(\alpha_2 - \alpha_4)}.$$

Additionally α_1, α_2, α_3 and α_4 are the roots of the polynomial equation

$$\Theta^4 + \frac{\xi_3}{\xi_4}\Theta^3 + \frac{\xi_2}{\xi_4}\Theta^2 + \frac{\xi_1}{\xi_4}\Theta + \frac{\xi_0}{\xi_4} = 0. \tag{79}$$

Substituting the solutions (74)-(78) into (9) and (17), we get

$$u(x,t) = \arccos\left[\frac{1}{2}\frac{\left(\tau_0+\tau_1\alpha_1 \pm \frac{\tau_1 A}{x\pm\sqrt{\frac{2\xi_4-\zeta_0\tau_1^2}{2\xi_4}}t-\eta_0}\right)^2+1}{\tau_0+\tau_1\alpha_1 \pm \frac{\tau_1 A}{x\pm\sqrt{\frac{2\xi_4-\zeta_0\tau_1^2}{2\xi_4}}t-\eta_0}}\right], \tag{80}$$

$$u(x,t) = \arccos\left[\frac{1}{2}\frac{\left(\tau_0+\tau_1\alpha_1+\frac{4A^2(\alpha_2-\alpha_1)\tau_1}{4A^2-\left[(\alpha_1-\alpha_2)\left(x\pm\sqrt{\frac{2\xi_4-\zeta_0\tau_1^2}{2\xi_4}}t-\eta_0\right)\right]^2}\right)^2+1}{\tau_0+\tau_1\alpha_1+\frac{4A^2(\alpha_2-\alpha_1)\tau_1}{4A^2-\left[(\alpha_1-\alpha_2)\left(x\pm\sqrt{\frac{2\xi_4-\zeta_0\tau_1^2}{2\xi_4}}t-\eta_0\right)\right]^2}}\right], \tag{81}$$

$$u(x,t) = \arccos\left[\frac{1}{2}\frac{\left(\tau_0+\tau_1\alpha_2+\frac{(\alpha_2-\alpha_1)\tau_1}{\exp\left[\frac{\alpha_1-\alpha_2}{A}\left(x\pm\sqrt{\frac{2\xi_4-\zeta_0\tau_1^2}{2\xi_4}}t-\eta_0\right)\right]-1}\right)^2+1}{\tau_0+\tau_1\alpha_2+\frac{(\alpha_2-\alpha_1)\tau_1}{\exp\left[\frac{\alpha_1-\alpha_2}{A}\left(x\pm\sqrt{\frac{2\xi_4-\zeta_0\tau_1^2}{2\xi_4}}t-\eta_0\right)\right]-1}}\right], \tag{82}$$

$$u(x,t) = \arccos\left[\frac{1}{2}\frac{\left(\tau_0+\tau_1\alpha_1+\frac{(\alpha_1-\alpha_2)\tau_1}{\exp\left[\frac{\alpha_1-\alpha_2}{A}\left(x\pm\sqrt{\frac{2\xi_4-\zeta_0\tau_1^2}{2\xi_4}}t-\eta_0\right)\right]-1}\right)^2+1}{\tau_0+\tau_1\alpha_1+\frac{(\alpha_1-\alpha_2)\tau_1}{\exp\left[\frac{\alpha_1-\alpha_2}{A}\left(x\pm\sqrt{\frac{2\xi_4-\zeta_0\tau_1^2}{2\xi_4}}t-\eta_0\right)\right]-1}}\right], \tag{83}$$

$$u(x,t) = \arccos\left[\frac{1}{2}\frac{\left(\tau_0+\tau_1\alpha_1-\frac{2(\alpha_1-\alpha_2)(\alpha_1-\alpha_3)\tau_1}{2\alpha_1-\alpha_2-\alpha_3+(\alpha_3-\alpha_2)\cosh\left(\frac{M}{A}\eta\right)}\right)^2+1}{\tau_0+\tau_1\alpha_1-\frac{2(\alpha_1-\alpha_2)(\alpha_1-\alpha_3)\tau_1}{2\alpha_1-\alpha_2-\alpha_3+(\alpha_3-\alpha_2)\cosh\left(\frac{M}{A}\eta\right)}}\right), \tag{84}$$

where $M = \sqrt{(\alpha_1-\alpha_2)(\alpha_1-\alpha_3)}$, $\eta = \left(x \pm \sqrt{\frac{2\xi_4-\zeta_0\tau_1^2}{2\xi_4}}t-\eta_0\right)$.

$$u(x,t) = \arccos\left[\frac{1}{2}\frac{\left(\tau_0+\tau_1\alpha_2+\frac{(\alpha_1-\alpha_2)(\alpha_4-\alpha_2)\tau_1}{(\alpha_1-\alpha_4)\mathbf{sn}^2(\varphi,l^2)+\alpha_4-\alpha_2}\right)^2+1}{\tau_0+\tau_1\alpha_2+\frac{(\alpha_1-\alpha_2)(\alpha_4-\alpha_2)\tau_1}{(\alpha_1-\alpha_4)\mathbf{sn}^2(\varphi,l^2)+\alpha_4-\alpha_2}}\right], \tag{85}$$

where $\varphi = \frac{\sqrt{(\alpha_1-\alpha_3)(\alpha_2-\alpha_4)}}{2A}\left(x \pm \sqrt{\frac{2\xi_4-\zeta_0\tau_1^2}{2\xi_4}}t - \eta_0\right)$, $l^2 = \frac{(\alpha_2-\alpha_3)(\alpha_1-\alpha_4)}{(\alpha_1-\alpha_3)(\alpha_2-\alpha_4)}$.

Remark 3. The obtained solutions (80)-(85) by use of the recommended method for Eq. (68) have been controlled by Mathematica package program. According to our research, the rational function, the optical singular soliton, dark optical soliton, bright optical soliton, and the Jacobi elliptic function solutions attained by use of the method defined in the research article are not shown in prior literature. The optical solutions obtained here are the new solutions of the equation Eq. (68).

Case 2

If we choose $\epsilon = 1$, $\delta = 1$ and $\theta = 5$, then

$$(v')^2 = \frac{\tau_1^2(\xi_5\Theta^5 + \xi_4\Theta^4 + \xi_3\Theta^3 + \xi_2\Theta^2 + \xi_1\Theta + \xi_0)}{\zeta_0 + \zeta_1\Theta}, \tag{86}$$

$$v'' = \frac{\tau_1\left[(\zeta_0+\zeta_1\Theta)\left(5\xi_5\Theta^4+4\xi_4\Theta^3+3\xi_3\Theta^2+2\xi_2\Theta+\xi_1\right)-\zeta_1\left(\xi_5\Theta^5+\xi_4\Theta^4+\xi_3\Theta^3+\xi_2\Theta^2+\xi_1\Theta+\xi_0\right)\right]}{2(\zeta_0+\zeta_1\Theta)^2}, \tag{87}$$

where $\xi_5 \neq 0$, $\zeta_1 \neq 0$. In turn, solving the system of algebraic equations (13) yields the following results:

$$\xi_0 = \xi_0,\ \xi_1 = \xi_1, \xi_2 = \xi_2, \xi_3 = \frac{2\tau_0\tau_1\left(4\xi_2\tau_0^2(\tau_0^4-1)+\tau_1\left(2\xi_0\tau_1(5\tau_0^4+3)-\xi_1\tau_0(7\tau_0^4+1)\right)\right)}{3(\tau_0^4-1)^2},\ \zeta_0 = \zeta_0,$$

$$\xi_4 = \frac{\tau_1^2\left(7\xi_2\tau_0(\tau_0^4-1)+\tau_1\left(2\xi_1\tau_0(1-8\tau_0^4)+\xi_0\tau_1(25\tau_0^4+3)\right)\right)}{3(\tau_0^4-1)^2},\ \xi_5 = \frac{\tau_1^3\left(2\xi_2\tau_0(\tau_0^4-1)+\tau_1\left(\xi_1(1-5\tau_0^4)+8\xi_0\tau_0^3\tau_1\right)\right)}{3(\tau_0^4-1)^2},$$

$$\tau_0 = \tau_0, \tau_1 = \tau_1,\ \zeta_1 = \frac{\zeta_0\left(2\xi_2\tau_0(\tau_0^4-1)+\tau_1\left(\xi_1(1-5\tau_0^4)+8\xi_0\tau_0^3\tau_1\right)\right)}{(\tau_0^4-1)^2(\xi_1\tau_0-2\xi_0\tau_1)},\ c = \pm\sqrt{\frac{\tau_1(\xi_1\tau_0-2\xi_0\tau_1)-\zeta_0(\tau_0^4-1)}{\tau_1(\xi_1\tau_0-2\xi_0\tau_1)}}.$$

Substituting these consequences into Eqs. (9) and (14), we achieve

$$\pm(\eta-\eta_0) = \sqrt{\frac{3\zeta_0}{\tau_1^3(\xi_1\tau_0-2\xi_0\tau_1)}}\int\sqrt{\frac{\Theta+\frac{\zeta_0}{\zeta_1}}{\Theta^5+\frac{\xi_4}{\xi_5}\Theta^4+\frac{\xi_3}{\xi_5}\Theta^3+\frac{\xi_2}{\xi_5}\Theta^2+\frac{\xi_1}{\xi_5}\Theta+\frac{\xi_0}{\xi_5}}}d\Theta. \tag{88}$$

Integrating Eq. (88), we attain the following exact approximate solutions to the Eq. (68).

When $\Phi(\Theta) = (\Theta-\alpha_1)^5$, we get

$$\pm(\eta-\eta_0) = -\frac{2A}{3\sqrt{\zeta_1}(\zeta_0+\zeta_1\alpha_1)}\left(\frac{\zeta_0+\zeta_1\Theta}{\Theta-\alpha_1}\right)^{\frac{3}{2}}. \tag{89}$$

If we choose $\Phi(\Theta) = (\Theta-\alpha_1)^4(\Theta-\alpha_2)$ and $\alpha_1 > \alpha_2$, then we find

$$\pm(\eta-\eta_0) = \frac{-A}{\alpha_1-\alpha_2}\left[\frac{(\zeta_0+\zeta_1\alpha_2)}{2\sqrt{\zeta_1(\alpha_1-\alpha_2)(\zeta_0+\zeta_1\alpha_1)}}\ln|K(\Theta)| + \frac{1}{\Theta-\alpha_1}\sqrt{\frac{(\zeta_0+\zeta_1\Theta)(\Theta-\alpha_2)}{\zeta_1}}\right], \tag{90}$$

where

$$K(\Theta) = \frac{\Theta-\alpha_1}{(\zeta_0+2\zeta_1\alpha_1-\zeta_1\alpha_2)\Theta+\zeta_0(\alpha_1-2\alpha_2)-\zeta_1\alpha_2\alpha_1+2\sqrt{(\zeta_0+\zeta_1\Theta)(\zeta_0+\zeta_1\alpha_1)(\Theta-\alpha_2)(\alpha_1-\alpha_2)}}. \tag{91}$$

When $\Phi(\Theta) = (\Theta-\alpha_1)^3(\Theta-\alpha_2)^2$ and $\alpha_1 > \alpha_2$, we attain

$$\pm(\eta-\eta_0) = \frac{-2A}{(\alpha_1-\alpha_2)}\sqrt{\frac{\zeta_0+\zeta_1\Theta}{\zeta_1(\Theta-\alpha_1)}} - \frac{2A}{(\alpha_1-\alpha_2)^{\frac{3}{2}}}\sqrt{\frac{\zeta_0+\zeta_1\alpha_2}{\zeta_1}}\arctan\left[\sqrt{\frac{(\Theta-\alpha_1)(\zeta_0+\zeta_1\alpha_2)}{(\alpha_1-\alpha_2)(\zeta_0+\zeta_1\Theta)}}\right]. \tag{92}$$

If we get $\Phi(\Theta) = (\Theta - \alpha_1)^2(\Theta - \alpha_2)^2(\Theta - \alpha_3)$ and $\alpha_1 > \alpha_2 > \alpha_3$, then we obtain

$$\pm(\eta - \eta_0) = \frac{-A}{(\alpha_1 - \alpha_3)\sqrt{\zeta_1}} \left[Y \ln |P(\Theta)| + Z \ln |R(\Theta)| \right], \tag{93}$$

where

$$Y = \sqrt{\frac{\zeta_0+\zeta_1\alpha_2}{\alpha_2-\alpha_3}}, \tag{94}$$

$$P(\Theta) = \frac{\alpha_2-\Theta}{2\sqrt{(\zeta_0+\zeta_1\Theta)(\zeta_0+\zeta_1\alpha_2)(\Theta-\alpha_3)(\alpha_2-\alpha_3)}+\zeta_0(\alpha_2-2\alpha_3)-\zeta_1\alpha_2\alpha_3+(\zeta_0+2\zeta_1\alpha_2-\zeta_1\alpha_3)\Theta}, \tag{95}$$

$$Z = \sqrt{\frac{\zeta_0+\zeta_1\alpha_1}{\alpha_1-\alpha_3}}, \tag{96}$$

$$R(\Theta) = \frac{2\sqrt{(\zeta_0+\zeta_1\Theta)(\zeta_0+\zeta_1\alpha_1)(\Theta-\alpha_3)(\alpha_1-\alpha_3)}+\zeta_0(\alpha_1-2\alpha_3)-\zeta_1\alpha_1\alpha_3+(\zeta_0+2\zeta_1\alpha_1-\zeta_1\alpha_3)\Theta}{\Theta-\alpha_2}. \tag{97}$$

When $\Phi(\Theta) = (\Theta - \alpha_1)^3(\Theta - \alpha_2)(\Theta - \alpha_3)$ and $\alpha_1 > \alpha_2 > \alpha_3$, then we find

$$\pm(\eta - \eta_0) = \frac{-2A}{\alpha_1 - \alpha_3}\sqrt{\frac{\zeta_0 + \zeta_1\alpha_3}{\zeta_1(\alpha_1 - \alpha_2)}} E(\varphi, l), \tag{98}$$

where

$$E(\varphi, l) = \int_0^{\varphi} \sqrt{1 - l^2 \sin^2 \psi} d\psi,\ \varphi = \arcsin \sqrt{\frac{(\Theta-\alpha_3)(\alpha_2-\alpha_1)}{(\Theta-\alpha_1)(\alpha_2-\alpha_3)}},\ l^2 = \frac{(\alpha_3-\alpha_2)(\zeta_0+\zeta_1\alpha_1)}{(\alpha_1-\alpha_2)(\zeta_0+\zeta_1\alpha_3)}. \tag{99}$$

If we acquire $\Phi(\Theta) = (\Theta - \alpha_1)^2(\Theta - \alpha_2)(\Theta - \alpha_3)(\Theta - \alpha_4)$ and $\alpha_1 > \alpha_2 > \alpha_3 > \alpha_3$, then we discover

$$\pm(\eta - \eta_0) = \frac{2A(\alpha_2-\alpha_4)}{(\alpha_1-\alpha_2)\sqrt{\zeta_1(\alpha_2-\alpha_3)(\zeta_0+\zeta_1\alpha_4)}} \left(\frac{\zeta_0+\zeta_1\Theta}{\alpha_1-\alpha_4}\pi(\varphi, n, l) - \frac{\zeta_0+\zeta_1\alpha_2}{\alpha_2-\alpha_4}F(\varphi, l) \right), \tag{100}$$

where

$$A = \sqrt{\frac{3\zeta_0}{\tau_1^3(\xi_1\tau_0-2\xi_0\tau_1)}},\ \pi(\varphi, n, l) = \int_0^{\varphi} \frac{d\psi}{(1+n\sin^2\psi)\sqrt{1-l^2\sin^2\psi}}, \tag{101}$$

and

$$\varphi = \arcsin \sqrt{\frac{(\Theta-\alpha_4)(\alpha_3-\alpha_2)}{(\Theta-\alpha_2)(\alpha_3-\alpha_4)}},\ l^2 = \frac{(\alpha_4-\alpha_3)(\zeta_0+\zeta_1\alpha_2)}{(\alpha_2-\alpha_3)(\zeta_0+\zeta_1\alpha_4)},\ n = -\frac{(\alpha_1-\alpha_2)(\alpha_3-\alpha_4)}{(\alpha_2-\alpha_3)(\alpha_1-\alpha_4)}. \tag{102}$$

4 Conclusions and Remarks

In this study, we have constituted the optical soliton solutions of the sine-Gordon and generalized sine-Gordon equations by use of the extended trial equation method. The obtained optical soliton solutions were found in terms of trigonometric, hyperbolic, rational, elliptic, and Jacobi elliptic functions. Therefore, it is indicated that some solutions in the literature are specific cases of the attained solutions. At the same time, the accuracy of the attained solutions has been controlled by the use of Mathematica. Moreover, the physical phenomena for the attained solutions are investigated by adding two and three dimensional plots of the optical soliton solutions to this study. We show that the extended trial equation method is an influential mathematical instrument for solving nonlinear differential equations.

5 Declarations

Funding

This research received no external funding.

Competing Interests

The authors declare that they have no competing interests.

Ethical Approval

Not applicable.

Authors' Contributions

All authors contributed equally. All the authors read and approved the final manuscript.

Availability Data and Materials

Not applicable.

Acknowledgements

The authors The authors would like to thank the anonymous reviewer for his/her comments.

References

[1] D.J. Benney and Alan C. Newell, *The propagation of nonlinear wave envelopes*, Journal of Mathematics and Physics **46** (1967), 133-139, doi.org/10.1002/sapm1967461133.

[2] M. Gros, Nguyen Van Tuan, P. Bertrand and G. Baumann, *Damping of solitons in plasmas*, Physics Letters A **71** (1979), no.2-3, 221-223, doi.org/10.1016/0375-9601(79)90168-3.

[3] Akira Hasegawa and Yuji Kodama, *Solitons in optical communications*, Oxford, 1995.

[4] Mark J. Ablowitz and Harvey Segur, *Soliton and the inverse scattering transformation*, PA: SIAM, Philadelphia, 1981.

[5] Deng-Shan Wang, Xing-Hua Hu and W.M. Liu, *Localized nonlinear matter waves in two-component Bose-Einstein condensates with time and space-modulated nonlinearities*, Physics Review A **82** (2010), 023612, doi.org/10.1103/PhysRevA.82.023612.

[6] A. R. Alharbi, M. B. Almatrafi and M. A. E. Abdelrahman, *An extended jacobian elliptic function expansion approach to the generalized fifth order KdV equation*, Physical Mathematics **10(4)** (2019), 1-5.

[7] Yusuf Gurefe and Emine Misirli, *Exp-function method for solving nonlinear evolution equations with higher order nonlinearity*, Computers and Mathematics with Applications **61** (2011), no.8, 2025-2030, doi.org/10.1016/j.camwa.2010.08.060.

[8] M. Ali Akbar and Norhashidah Hj. Mohd. Ali, *The improved F-expansion method with Riccati equation and its applications in mathematical physics*, Cogent Mathematics **4** (2017), no.1, Article ID 1282577, 1-19, doi.org/10.1080/23311835.2017.1282577.

[9] N. H. Aljahdaly, *Some applications of the modified (G'/G^2)-expansion method in mathematical physics*, Results in Physics **13** (2019), Article ID 102272, 1-7, doi.org/10.1016/j.rinp.2019.102272.

[10] Yusuf Gurefe and Emine Misirli, *New variable separation solutions of two-dimensional Burgers system*, Applied Mathematics and Computation **217** (2011), no.22, 9189-9197, doi.org/10.1016/j.amc.2011.03.157.

[11] Elsayed Zayed, Mahmoud Abdelaziz and Mushrifa Elmalky, *Enhanced (G'/G)-expansion method and applications to the (2+1)D typical Breaking soliton and Burgers equations*, Journal of Advanced Mathematical Studies **4** (2011), no.2, 109-122.

[12] Nikolay A. Kudryashov, *One method for finding exact solutions of nonlinear differential equations*, Communications in Nonlinear Science and Numerical Simulation **17** (2012), no.6, 2248-2253, doi.org/10.1016/j.cnsns.2011.10.016.

[13] Yusuf Pandir, Yusuf Gurefe and Emine Misirli, *A new approach to Kudryashov's method for solving some nonlinear physical models*, International Journal of Physical Sciences **7** (2012), no.21, 2860-2866, doi.org/10.5897/IJPS12.071.

[14] Yusuf Gurefe, Yusuf Pandir and Emine Misirli, *New exact solutions of stochastic KdV equation*, Applied Mathematical Sciences **6** (2012), no.65, 3225-3234.

[15] Juan Yang and Qingjiang Feng, *The soliton solutions and evolution of the (2+1)-dimensional dissipative long wave equation*, Results in Physics **21** (2021), Article ID 103794, 1-7, doi.org/10.1016/j.rinp.2020.103794.

[16] Yukun Zhao, Yujie Chen, Jun Dai, Ying Wang and Wei Wang, *Bright soliton solution of (1+1)-dimensional quantum system with power-law dependent nonlinearity*, Advances in Mathematical Physics **2019** (2019), Article ID 8264848, 1-5, doi.org/10.1155/2019/8264848.

[17] Cheng-Shi Liu, *Trial equation method and its applications to nonlinear evolution equations*, Acta Phyica Sinica **54** (2005), no.6, 2505-2509, doi.org/10.7498/aps.54.2505.

[18] Cheng-Shi Liu, *Trial equation method to nonlinear evolution equations with rank inhomogeneous: mathematical discussions and its applications*, Communications in Theoretical Physics **45** (2006), no.2, 219-223, doi.org/10.1088/0253-6102/45/2/005.

[19] Cheng-Shi Liu, *A new trial equation method and its applications*, Communications in Theoretical Physics **45** (2006), no.3, 395-397, doi.org/10.1088/0253-6102/45/3/003.

[20] Damin Cao and Lijuan Du, *The classification of the single traveling wave solutions to (1 + 1) dimensional Gardner equation with variable coefficients*, Advances in Difference Equations **2019** (2019), Article ID 121, 1-15, doi.org/10.1186/s13662-019-2061-0.

[21] Cheng-Shi Liu, *Trial equation method based on symmetry and applications to nonlinear equations arising in mathematical physics*, Foundations of Physics **41** (2011), 793-804, doi.org/10.1007/s10701-010-9521-4.

[22] Yusuf Pandir, Yusuf Gurefe, Uğur Kadak and Emine Misirli, *Classifications of exact solutions for some nonlinear partial differential equations with generalized evolution*, Abstract and Applied Analysis **2012** Article ID 478531 (2012), 1-16, doi.org/10.1155/2012/478531.

[23] Cheng-Shi Liu, *Applications of complete discrimination system for polynomial for classifications of travelling wave solutions to nonlinear differential equations*, Computer Physics Communications **181** (2010), no.2, 317-324, doi.org/10.1016/j.cpc.2009.10.006.

[24] Li Yang, *Application of trial equation method for solving the Getmanou equation*, Applied Mathematics **5(10)** (2014), 1462-1473, doi.org/10.4236/am.2014.510139.

[25] C.Y. Jun, *Classification of traveling wave solutions to the Vakhnenko equations*, Computers and Mathematics with Applications **62** (2011), no.10, 3987-3996, doi.org/10.1016/j.camwa.2011.09.060.

[26] Yang Liu, *Exact solutions to nonlinear Schrödinger equation with variable coefficients*, Applied Mathematics and Computation **217** (2011), no.12, 5866-5869, doi.org/10.1016/j.amc.2010.12.072.

[27] L. Yang, *Trial equation method for solving the improved Boussinesq equation*, Advances in Pure Mathematics **4(2)** (2014), Article ID 43372, 1-6, doi.org/10.4236/apm.2014.42007.

[28] Cheng Yan Jun, *Classification of traveling wave solutions to the modified form of the Degasperis-Procesi equation*, Mathematical and Computer Modelling **56** (2012), no.1-2, 43-48, doi.org/10.1016/j.mcm.2011.12.021.

[29] Yusuf Gurefe, Emine Misirli, Abdullah Sonmezoglu and Mehmet Ekici, *Extended trial equation method to generalized nonlinear partial differential equations*, Applied Mathematics and Computation **219** (2012), no.10, 5253-5260, doi.org/10.1016/j.amc.2012.11.046.

[30] Abdul-Majid Wazwaz, *The tanh method: exact solutions of the sine-Gordon and the sinh-Gordon equations*, Applied Mathematics and Computation **167** (2005), no.2, 1196-1210, doi.org/10.1016/j.amc.2004.08.005.

[31] Abdul-Majid Wazwaz, *Exact solutions for the generalized sine-Gordon and the generalized sinh-Gordon equations*, Chaos, Solitons and Fractals **28(1)** (2006), 127-135, doi.org/10.1016/j.chaos.2005.05.017.

[32] Yoshimasa Matsuno, *A direct method for solving the generalized sine-Gordon equation*, Journal of Physics A: Mathematical and Theoretical **43** (2010), no.10, Article ID 105204, doi.org/10.1088/1751-8113/43/10/105204.

[33] Tuncay Aktosun, Francesco Demontis and Cornelis van der Mee, *Exact solutions to the sine-Gordon equation*, Journal of Mathematical Physics **51** (2010), 1235211, 1-43, doi.org/10.1063/1.3520596.

[34] G.L. Lamb, Jr., *Elements of Soliton Theory*, Wiley, New York, (1980).

[35] C. Rogers and W.K. Schief, *Backlund and Darboux transformations*, Cambridge University Press, Cambridge, (2002).